Appendix A: The conditional Gaussian process prior.

The conditional GP prior $f(S)$ can be derived first by applying the techniques for obtaining the conditional normal distribution from a joint normal distribution. In the standard multivariate normal setting, when x_1 and x_2 are jointly normal with mean $[\mu_1, \mu_2]^T$ and covariance Σ, the distribution for x_2 given x_1 is

(A.1) $$\mu_{2|1} = \mu_2 + \Sigma_{2,1}\left(\Sigma_{1,1}\right)^{-1}(x_1 - \mu_1)$$

$$\Sigma_{2|1} = \Sigma_2 - \Sigma_{2,1}\left(\Sigma_{1,1}\right)^{-1}\Sigma_{2,1}{}^T,$$

where $\Sigma_{i,j}$ denotes the partition of Σ corresponding covariance among i and j, e.g., $\Sigma_{1,2}$, represents the covariances among elements of vectors x_1 and x_2 (e.g., Hogg et al. 2004).

To make use of this we construct a joint GP prior:

(A.2) $$\begin{bmatrix} f(S) \\ f(0) \end{bmatrix} \sim \mathrm{GP}\left[\begin{pmatrix} \mu(S) \\ \mu(0) \end{pmatrix}, \begin{pmatrix} \Sigma(S,S') & \Sigma(S,0) \\ \Sigma(S,0)^T & \Sigma(0,0') \end{pmatrix}\right].$$

$\Sigma(S,S')$ is a covariance function for $f(S)$, which is given by $\Sigma(S,S') = \tau^2 \exp[-\phi/ |(S-S')/\max(S)|^2]$. $\Sigma(S,0)$ is a covariance between $f(S)$ and $f(0)$, which is $\Sigma(S,0) = \tau^2 \exp[-\phi|S/\max(S)|^2]$. $\Sigma(0,0')$ is a variance for an univariate Gaussian process model $f(0)$, which is $\Sigma(0,0') = \tau^2$. With Eqs. A.1 and A.2, we can derive the conditional GP prior:

(A.3) $$f(S)|f(0) = \ln\alpha \sim \mathrm{GP}[\mu_c, \Sigma_c]$$

where

(A.4) $$\mu_c = \mu(S) + \Sigma(S,0)[(0,0')]^{-1}[\ln\alpha - \mu(0)] = \mu(S)$$

(A.5) $$\Sigma_c = \Sigma(S,S') - \Sigma(S,0')[\Sigma(0,0')]^{-1}\Sigma(S,0')^T$$

$$= \tau^2 \exp\left[-\phi\left|\frac{S-S'}{\max(S)}\right|^2\right] - \tau^2 \exp\left\{-\phi\left[\frac{S^2 + S'^2}{\max(S)^2}\right]\right\}.$$

Literature cited

Hogg, R.V., J. W. McKean, and A. Craig. 2004. Introduction to mathematical statistics. Prentice Hall (Sixth edition), Upple Saddle River, New Jersey, USA.