

Appendix A. Yodzis-Innes (YI) model. Ecological Archives A024-217-A1.

From: “Limitations of extrapolating toxic effects on reproduction to the population level”

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## I. YI consumer-resource model and equilibria

The dynamics of a consumer population's biomass,  $C$ , feeding on a shared resource,  $R$ , is given by:

$$\frac{dR}{dt} = \rho(R_{\max} - R) - MC \frac{R}{H + R} \quad \text{Eq. A.1}$$

$$\frac{dC}{dt} = \frac{R}{H + R} M\sigma C - TC - \mu C \quad \text{Eq. A.2}$$

where  $\rho$  is the resource turnover rate, and  $R_{\max}$  is the maximum density the resource can reach in the absence of consumers.  $M$  and  $H$  are the maximum mass-specific feeding rate and the half-saturation coefficient of the consumer on the resource, and  $\sigma$  is the assimilation efficiency. The consumer population loses biomass at a rate proportional to its biomass via maintenance costs and mortality where  $T$  and  $\mu$  represent the mass specific maintenance loss rate and the mortality rate.

At equilibrium the consumer's population mass-specific net biomass production,  $P$  (rate of assimilating biomass minus maintenance rate) must be balanced with losses through mortality:

$$\frac{R^*}{H + R^*} M\sigma - T = P^* = \mu \quad \text{Eq. A.3}$$

The non-zero equilibria of the resource and consumer biomass are:

$$R^* = \frac{H(T + \mu)}{M\sigma - T + \mu} \quad \text{Eq. A.4}$$

$$C^* = \frac{\sigma\rho(R_{\max} - R^*)}{T + \mu} \quad \text{Eq. A.5}$$

## II. YI individual model and derivation of $F(t_{obs})$

The YI model can be derived from assumptions on *individual* consumers. *Juveniles* are born with a mass  $\omega_B$ , are assumed to feed at a rate proportional to their mass and to have a type 2 functional response with half saturation constant  $H$  independent of mass. Maintenance costs for each individual are likewise assumed proportional to mass. All net-mass-specific biomass production,  $P$ , is allocated to growth until the organism achieves a mass  $\omega_A$ . Thus the mass of an individual juvenile changes in accordance with the differential equation:

$$\frac{dW}{dt} = WP = W \left( \frac{R^*}{H + R^*} M\sigma - T \right). \quad \text{Eq. A.6}$$

Upon reaching maturity, individuals stop growing and allocate all subsequent net biomass production to reproduction. The reproduction rate in terms of offspring,  $\beta$ , can then be calculated by the product of mass specific biomass production and the mass of an adult divided by the mass of a neonate:

$$\beta = P \frac{\omega_A}{\omega_B} = \left( \frac{R}{R + H} M\sigma - T \right) \frac{\omega_A}{\omega_B} \quad \text{Eq. A.7}$$

Methodology from structured population theory can be used to derive the population equations (A.1, A.2) from these individual equations (Nisbet et al. 1997).

We used the individual model in this paper to describe standardized toxicity tests for the effects of a stressor on individual reproduction that are conducted at or near, *ad libitum* food so

that the scaled functional response,  $f = \frac{R}{R + H}$ , goes to unity. The cumulative number of

offspring an individual produces in a reproduction test of duration,  $t_{obs}$ , can be calculated as the product of reproduction rate and the amount of time in the experiment the individual was an adult. The time needed to reach adult mass,  $\omega_A$  at *ad libitum* food can be found by solving Eq.

A.6:

$$t_A = P^{-1} \ln \frac{\omega_A}{\omega_B} \quad \text{Eq. A.8}$$

Thus at the cumulative reproductive output in a reproduction tests is given by:

$$F(t_{obs}) = P \frac{\omega_A}{\omega_B} (t_{obs} - t_A) \quad \text{Eq. A.9}$$

Substituting in the expressions for  $P$  and  $t_A$  and simplifying gives the equation given in Table 1

of the manuscript:

$$F(t_{obs}) = \frac{\omega_A}{\omega_B} \left[ (M\sigma - T)t_{obs} - \ln \frac{\omega_A}{\omega_B} \right] \quad \text{Eq. A.10}$$

### III. Parameters used for YI model

Table A1

| <i>Model parameters and compound variables</i> |  |  |        |
|--|--|--|--------|
| Symbol   | Description                                    | Value (units)                                  | Source |
| $R_{\max}$                                     | Maximum resource biomass                       | $2\text{-}20H$ (mg/L)                          | -      |
| $\rho$   | Resource dilution rate                         | $0.1$ (day <sup>-1</sup> )                     | 1      |
| $\sigma$                                       | Conversion efficiency                          | $0.5$ (-)                                      | 1      |
| $\omega_A$                                     | Mass at maturity                               | $8.2 \times 10^{-5}$ (grams)                   | 2      |
| $\omega_B$                                     | Mass at birth                                  | $1.3 \times 10^{-5}$ (grams)                   | 2      |
| $\mu$  | Background mortality rate                      | $0.0015 \omega_A^{-0.25}$ (day <sup>-1</sup> ) | 1      |
| $M$  | Maximum mass-specific feeding rate             | $0.1 \omega_A^{-0.25}$ (day <sup>-1</sup> )    | 1      |
| $T$  | Mass-specific maintenance rate                 | $0.01 \omega_A^{-0.25}$ (day <sup>-1</sup> )   | 1      |
| $H$  | Half-saturation coefficient                    | $3.0$ (mg/L)                                   | 1      |
| $t_{\text{obs}}$                               | Observation period for individual reproduction | $21$ (days)                                    | OECD   |

Sources: 1. Default mass specific values given in box 3.3 in de Roos and Persson (2013),  
 2. Newborn and adult masses taken from table 4.1 in Nisbet et al. (1997). Values for *D. pulex* are used and converted from mgC to wet weight assuming 0.42 ratio of carbon weight to dry weight, and 0.2 dry weight to wet weight ratio (box 3.3. of de Roos and Perrson 2013)

### IV. Effects of stress on population abundance and size-structure in the YI model

The equilibrium population age distribution  $f(a)$  is defined by setting  $f(a)da$  to be the number of individuals in the infinitesimal age interval  $a \rightarrow a + da$ . Because mortality is independent of age and mass, this distribution follows a simple exponential decay

$$f(a) = f(0)e^{-\mu a} \quad \text{with } f(0) = R_J \quad \text{Eq. A.11}$$

and  $R_J$  is the total rate of reproduction (neonates/time).

The mass,  $W(a)$ , of an individual aged  $a$  at equilibrium grows exponentially at a rate determined by the net production per unit mass,  $P$ , so

$$W(a) = \omega_B \exp(Pa), \quad \text{Eq. A.12}$$

implying that the age at first reproduction,  $a_m$ , (when mass =  $\omega_A$ ) is

$$a_m = P^{-1} \ln(\omega_A / \omega_B). \quad \text{Eq. A.13}$$

The population of juveniles at equilibrium is

$$N_J = \int_0^{a_m} f(a) da = \frac{R_J}{\mu} (1 - e^{-\mu a_m}), \quad \text{Eq. A.14}$$

And the adult population is

$$N_A = \frac{R_J e^{-\mu a_m}}{\mu}. \quad \text{Eq. A.15}$$

At equilibrium, the rate of net biomass production through growth and reproduction must be equal to the loss through mortality. Thus if  $C_J$  and  $C_A$  represent respectively the biomasses of juveniles and adults,

$$\underbrace{PC_J}_{\text{growth}} + \underbrace{PC_A}_{\text{reproduction}} = \underbrace{\mu(C_J + C_A)}_{\text{mortality}} \quad \text{Eq. A.16}$$

implying that  $P = \mu$ .

This simplification lets us calculate the total population (adults + juveniles) at equilibrium as

$$N = N_J + N_A = \frac{R_J}{\mu} = \frac{R_J}{P}, \quad \text{Eq. A.17}$$

the equilibrium biomass of juveniles as

$$C_J = \int_0^{a_m} f(a)W(a)da = R\omega_B \int_0^{a_m} e^{-\mu a} e^{Pa} da = R_J \omega_B a_m = \frac{R_J \omega_B}{P} \ln \frac{\omega_M}{\omega_B}, \quad \text{Eq. A.18}$$

the equilibrium biomass of adults as

$$C_A = \omega_A N_A = \frac{\omega_A R_J e^{-\mu a_m}}{\mu} = \frac{\omega_A R_J e^{-Pa_m}}{P} = \frac{\omega_B R_J}{P}, \quad \text{Eq. A.19}$$

and the total biomass of the population,  $C$ , as

$$C = C_J + C_A = \frac{\omega_B R_J}{P} \left( \ln \frac{\omega_M}{\omega_B} + 1 \right). \quad \text{Eq. A.20}$$

From equations (A.17) and (A.20), the ratio

$$\frac{C}{N} = \omega_B \left( \ln \frac{\omega_M}{\omega_B} + 1 \right) \quad \text{Eq. A.21}$$

and thus depends only on the masses at birth and maturity. In our study of sublethal effects, these parameters are unaffected by stressors (they only impact  $P$  and  $R$  in the preceding analysis), so the effects of stress is to reduce equilibrium biomass and population in direct proportion.

## Literature cited

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