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Otso Ovaskainen. 2004. Habitat-specific movement parameters estimated using mark–recapture data and a diffusion model. *Ecology* 85:242-257.

Appendix A. Numerical solutions by the finite element method.

The finite element method (FEM) is a standard tool for finding numerical solutions to partial differential equations (Zienkiewicz and Taylor 2000). As implied by its name, the method is based on dividing a domain Ω to a number of smaller subdomains, which are called elements. The shape of the most suitable elements depends on the geometry of the problem considered, but a majority of applications uses triangles in two dimensions and tetrahedrons in three dimensions. The basic assumption of the method is that the exact solution v may be approximated by a function v^*, which is assumed to be a linear combination of fixed shape functions, which are typically low-order polynomials within each element. In order to utilize the finite element method, the problem has to be transformed into a variational form. To do this, I first multiply both sides of Eq. 1 by an arbitrary continuous function u, and integrate over the domain Ω.

$$\int_{\Omega} v_t(\mathbf{x},t)\, u(\mathbf{x},t) = \sum_i \int_{\Omega_i} [D_i(t)\, \Delta v(\mathbf{x},t) - \mu_i(t)\, v(\mathbf{x},t)\,]\, u(\mathbf{x},t)\, d\mathbf{x}. \qquad \text{(A.1)}$$

Integrating by parts gives

$$\int_{\Omega} v_t(\mathbf{x},t)\, u(\mathbf{x},t) = \sum_i \int_{\partial\Omega_i} D_i(t)\, v_n(\mathbf{x},t)\, u(\mathbf{x},t)\, dS \qquad \text{(A.2)}$$

$$- \sum_i \int_{\Omega_i} [D_i(t)\, \nabla v(\mathbf{x},t) \cdot \nabla u(\mathbf{x},t) + \mu_i(t)\, v(\mathbf{x},t)\, u(\mathbf{x},t)\,]\, d\mathbf{x}.$$

As the flux $D_i(t)\, v_n(\mathbf{x},t)$ is assumed to be continuous through the interior boundaries and zero at ∂Ω, the boundary term in Eq. A.2 vanishes, and we obtain

$$\int_{\Omega} v_t(\mathbf{x},t)\, u(\mathbf{x},t) = -\sum_i \int_{\Omega_i} [D_i(t)\, \nabla v(\mathbf{x},t) \cdot \nabla u(\mathbf{x},t) + \mu_i(t)\, v(\mathbf{x},t)\, u(\mathbf{x},t)\,]\, d\mathbf{x}. \qquad (A.3)$$

Equation A.3 defines the weak or the variational formulation of the original problem. It is neither necessary nor possible to go deep in the mathematical detail here, but the essential point is that a function v that is sufficiently smooth and that satisfies the weak formulation for all u belonging to a suitable function space is also the solution to the original problem .

In order to discretize the problem, I let $\tilde{N}_j$ (j=1,…, m) denote a set of shape functions which are continuous in Ω. In order to take into account for the discontinuities through the interior boundaries Γ_i, I let N_j denote the shape functions defined by $N_j(\mathbf{x})=k(\mathbf{x})\, \tilde{N}_j(\mathbf{x})$, where $k(\mathbf{x})=k_i$ for $\mathbf{x} \in \Omega_i$. Furthermore, I denote the finite element approximation for v at time t by $v^*(\mathbf{x},t;\mathbf{y})=\Sigma_j\, a_j(t)\, N_j(\mathbf{x})$, where $\mathbf{a}(t)=\{a_j(t)\}$ is the coefficient vector. Then, assuming that v^* satisfies the weak formulation (Eq. A.3) for all $u=\tilde{N}_k$, it follows that **a** satisfies

$$\Sigma_i\, A^i \frac{\partial \mathbf{a}(t)}{\partial t} + \Sigma_i\, (D_i(t)\, B^i + \mu_i(t)\, A^i)\, \mathbf{a}(t) = A(t) \frac{\partial \mathbf{a}(t)}{\partial t} + K(t)\, \mathbf{a}(t) = 0, \qquad (A.4)$$

where A^i and B^i denote the matrices with elements

$$A^i_{jk} = \int_{\Omega_i} \tilde{N}_j(\mathbf{x})\, N_k(\mathbf{x})\, d\mathbf{x}, \qquad (A.5)$$

$$B^i_{jk} = \int_{\Omega_i} \nabla \tilde{N}_j(\mathbf{x}) \cdot \nabla N_k(\mathbf{x})\, d\mathbf{x}, \qquad (A.6)$$

and A(t) and K(t) denote the appropriate sums of the matrices. In order to discretize Eq. A.4 in time, I let Δt denote the time step and write $t=T\,\Delta t$ where T=0,1,…, so that $a_T=a(T\,\Delta t)$. Assuming that the matrices A(t) and K(t) are constants within each time step, simple single-step algorithms lead to the relation (Zienkiewicz and Taylor 2000)

$$\mathbf{a}_{T+1} = (A_T + \theta \Delta t\, K_T)^{-1}\, (A_T - (1-\theta)\, \Delta t\, K_T)\, \mathbf{a}_T. \qquad (A.7)$$

In Eq. A.7, $0 \le \theta \le 1$ is a weighting parameter, which I have set to $\theta = 2/3$ corresponding to the Galerkin scheme. Equation A.7 represents a system of linear equations, from which $\mathbf{a}_{T+1}$ may be solved given that $\mathbf{a}_T$ is known.

Literature cited

Zienkiewicz, O. C., and R. L. Taylor. 2000. The finite element method: Volume 1: the basis. Butterworth-Heinemann, Oxford, Massachusetts, USA.

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