

## *Ecological Archives* E085-010-A1

Appendix B. Asymptotic speed of invasion. Consider the linearization of (20),

$$A_{n+1} = A_n + R_0 A_n + R_1 K * A_n, \quad (\text{B.1})$$

which describes plant dynamics far in advance of the wave of invasion, where  $A_n \ll 1$ . We seek solutions to (B.1) of the form

$$A_n(x) \sim e^{-sx}, \quad \text{and} \quad A_n(x - c) = A_{n+1}, \quad (\text{B.2})$$

corresponding to an exponentially decaying wave which moves a distance  $c$  in each iteration. Let  $K$  be the Laplace distribution kernel with mean dispersal distance  $\alpha$ ,

$$K = \frac{1}{2\alpha} \exp\left[-\frac{|x|}{\alpha}\right]. \quad (\text{B.3})$$

Then

$$K * A_n = \frac{1}{2\alpha} \int_{-\infty}^{\infty} e^{-sy} e^{-\frac{|x-y|}{\alpha}} dy, \quad (\text{B.4})$$

and with the change of variables  $z = x - y$  we get

$$K * A_n = e^{-sx} \int_{-\infty}^{\infty} e^{sz} e^{-\frac{|z|}{\alpha}} dz = M(s), \quad (\text{B.5})$$

where  $M(s)$  is the moment-generating function for  $K$ . One can then calculate  $M(s) = (1 - \alpha^2 s^2)^{-1}$ . Substituting these results into (B.1), letting  $A_{n+1} \sim e^{-s(x-c)}$  and cancelling the common factor  $e^{-sx}$  gives a dispersion relation for the exponential decay constant,  $s$ , and the rate of propagation,  $c$ ,

$$e^{sc} = 1 + R_0 + R_1 \frac{1}{1 - \alpha^2 s^2} = 1 + R_0 + R_1 M(s). \quad (\text{B.6})$$

An additional condition for the minimum wave speed is that  $\frac{dc}{ds} = 0$ . Using this to simplify the derivative (in  $s$ ) of (B.6) gives

$$ce^{sc} = R_1 \frac{2\alpha^2 s}{(1 - \alpha^2 s^2)^2} = R_1 M'(s). \quad (\text{B.7})$$

Solving these two equations for  $c$  gives

$$c = \frac{R_1 M'(s)}{1 + R_0 + R_1 M(s)} = \frac{2R_1 \alpha^2 s}{(1 + R_0)(1 - \alpha^2 s^2)^2 + R_1(1 - \alpha^2 s^2)}. \quad (\text{B.8})$$

On the other hand, eliminating  $c$  from (B.6) and (B.7) gives

$$\exp \left[ \frac{2R_1 \alpha^2 s^2}{(1 + R_0)(1 - \alpha^2 s^2)^2 + R_1(1 - \alpha^2 s^2)} \right] = 1 + R_0 + R_1 \frac{1}{1 - \alpha^2 s^2}. \quad (\text{B.9})$$

Solutions to (B.8, B.9) give the expected rate of invasion for (B.1) and thence (20).

In the limit of small growth rates we may take  $R_1 = \epsilon r_1$ ,  $\epsilon \ll 1$  and  $R_0 = \epsilon r_0 = \epsilon \rho r_1$ , (and not to be confused with the  $\epsilon$  appearing in the previous appendix or main text). Equation (B.9) becomes

$$0 = \exp \left[ \frac{2\epsilon r_1 \alpha^2 s^2}{(1 + \epsilon r_0)(1 - \alpha^2 s^2)^2 + \epsilon r_1(1 - \alpha^2 s^2)} \right] - \left( 1 + \epsilon r_0 + \frac{\epsilon r_1}{1 - \alpha^2 s^2} \right). \quad (\text{B.10})$$

Introducing the regular perturbation approximation,

$$s = s_0 + \epsilon s_1 + \epsilon^2 s_2 + \dots, \quad (\text{B.11})$$

and expanding (B.10) as a Maclaurin series in  $\epsilon$  gives the expression

$$0 = \epsilon f_1(s_0, \alpha, \rho, r_1) + \epsilon^2 f_2(s_0, s_1, \alpha, \rho, r_1) + \epsilon^3 f_3(s_0, s_1, s_2, \alpha, \rho, r_1) + \dots. \quad (\text{B.12})$$

Here

$$f_1(s_0, \alpha, \rho, r_1) = r_1 \frac{\alpha^2 s_0^2 (3 + 2\rho) - \rho(1 + \alpha^4 s_0^4) - 1}{(\alpha^2 s_0^2 - 1)^2}, \quad (\text{B.13})$$

and

$$f_2(s_0, s_1, \alpha, \rho, r_1) = \frac{4r_1 s_0 \alpha^2}{(\alpha^2 s_0^2 - 1)^4} \left[ s_1 \left( 2\alpha^2 s_0^2 - 3\alpha^4 s_0^4 + 1 \right) + r_1 s_0 \left( 2(1 + \rho)\alpha^2 s_0^2 - 1 - \rho(1 + \alpha^4 s_0^4) \right) \right]. \quad (\text{B.14})$$

The functional form for  $f_3$  is not particularly enlightening. It is best determined using a symbolic manipulation package (like *Maple* or *Mathematica*), and is linear in  $s_2$ .

As  $\epsilon$  is thought to be a small parameter which varies from circumstance to circumstance, each of  $f_1, f_2, \dots$  must vanish independently. Solving the equation  $f_1 = 0$  for  $s_0$  (and choosing the

root corresponding to the minimum speed) gives

$$s_0 = \frac{\sqrt{3 + 2\rho - \sqrt{9 + 8\rho}}}{\alpha\sqrt{2\rho}}. \quad (\text{B.15})$$

Inserting the solution for  $s_0$  into  $f_2 = 0$  and solving for  $s_1$  gives the first correction term

$$s_1 = \frac{r_1 \sqrt{\rho} (3 + 2\rho - \sqrt{9 + 8\rho})^{\frac{3}{2}}}{\sqrt{2} \alpha (4\rho (-6 + \sqrt{9 + 8\rho}) + 9 (-3 + \sqrt{9 + 8\rho}))}. \quad (\text{B.16})$$

Substituting into Eq. B.8 and expanding gives an expression for  $c$ ,

$$c = 4 \epsilon \sqrt{2} \alpha R_1 \rho^{\frac{3}{2}} \frac{\sqrt{3 + 2\rho - \sqrt{9 + 8\rho}}}{(9 + 8\rho - 3\sqrt{9 + 8\rho})^2} + \quad (\text{B.17})$$

$$\epsilon^2 \frac{16 \alpha R_1^2 \rho^{\frac{5}{2}} \sqrt{6 + 4\rho - 2\sqrt{9 + 8\rho}} (27 + 33\rho + 8\rho^2 - (9 + 7\rho) \sqrt{9 + 8\rho})}{(-3 + \sqrt{9 + 8\rho})^4 (-9 - 8\rho + 3\sqrt{9 + 8\rho})} + \mathcal{O}(\epsilon^3).$$

Substituting  $\epsilon r_1 = R_1$  gives the asymptotic speed result (22) in the main text.

Since all of the expressions for  $c$  and  $s$  are smooth and analytic at  $\epsilon = 0$  the Maclaurin expansion used to generate the asymptotic expansion is guaranteed to converge for sufficiently small  $\epsilon$ . However, it must be remembered that the expansion is *joint* in the small parameters  $R_1, R_0 = \rho R_1$ , and consequently the convergence of the approximation is conditioned on the size of  $\rho$  as well. Consequently, the size of the error is reported more explicitly as  $\mathcal{O}(\epsilon^3(1 + \rho^3))$ , indicating the dependence of the error on the parameter  $\rho$ .