

Moorcroft, Paul R., and Alex Barnett. 2008. Mechanistic home range models and resource selection analysis: a reconciliation and unification. *Ecology* 89 :1112-1119.

Appendix A. Derivation of the Fokker-Planck equation for space-use.

Appendix

Derivation of the Fokker-Planck equation for space-use

Here we derive equation Eqs. (9)-(10) from Eq. (8), for the case of general redistribution kernel $k_\tau(x, x')$ ¹. First we note that in order for $u(x, t)$ to remain a probability density function, the kernel must satisfy

$$\int k_\tau(x, x') dx = 1, \quad \text{for all } x'. \quad (\text{A.1})$$

Recalling the definition $q := x - x'$, we now change the variables used to describe the kernel from (x, x') to (q, x') , by defining

$$\kappa_\tau(q, x') := k_\tau(x' + q, x'). \quad (\text{A.2})$$

The purpose of this is to enable us to hold q constant while expanding a Taylor series in x' . Writing Eq. (8) in terms of this new kernel, then changing integration variable from x' to q gives

$$\begin{aligned} u(x, t + \tau) &= \int \kappa_\tau(x - x', x') u(x', t) dx' \\ &= \int \kappa_\tau(q, x - q) u(x - q, t) dq \\ &= \int \left[\kappa_\tau(q, x) u(x, t) - q \frac{\partial}{\partial x} [\kappa_\tau(q, x) u(x, t)] + \frac{q^2}{2!} \frac{\partial^2}{\partial x^2} [\kappa_\tau(q, x) u(x, t)] \dots \right] dq. \end{aligned} \quad (\text{A.3})$$

The final step is achieved by considering the integrand as a function of two variables, q and $(x - q)$, and Taylor expanding this function with respect to the second variable about the value x , while holding the first constant.

Dividing (A.3) by τ , making use of $\int \kappa_\tau(q, x) dq = 1$ which is a re-statement of Eq. (A.1), and switching the order of differentiation and integration we get

$$\frac{u(x, t + \tau) - u(x, t)}{\tau} = -\frac{1}{\tau} \frac{\partial}{\partial x} \int q \kappa_\tau(q, x) dq u(x, t) + \frac{1}{2\tau} \frac{\partial^2}{\partial x^2} \int q^2 \kappa_\tau(q, x) dq u(x, t) \dots (\text{A.4})$$

Taking the limit of small time interval τ gives

$$\begin{aligned} \frac{\partial u(x, t)}{\partial t} &= -\frac{\partial}{\partial x} \left[\lim_{\tau \rightarrow 0} \frac{1}{\tau} \left(\int_{-\infty}^{\infty} q \kappa_\tau(q, x) dq \right) u(x, t) \right] \\ &+ \frac{\partial^2}{\partial x^2} \left[\lim_{\tau \rightarrow 0} \frac{1}{2\tau} \left(\int_{-\infty}^{\infty} q^2 \kappa_\tau(q, x) dq \right) u(x, t) \right] - \dots \end{aligned} \quad (\text{A.5})$$

¹This is essentially the derivation of a Fokker-Planck equation from a Chapman-Kolmogorov master equation via the Kramers-Moyal expansion – see Gardiner 1983.

The two integrals in the first and second terms are, respectively, the first and second moments of the kernel κ_τ . Considering the limit of small time-steps ($\tau \rightarrow 0$), and making the conventional assumptions that the first two moments scale with order τ and that higher order terms can be discarded, we arrive at Eqs. (9)-(10)

$$\frac{\partial u(x, t)}{\partial t} = -\frac{\partial}{\partial x} [c(x)u(x, t)] + \frac{\partial^2}{\partial x^2} [d(x)u(x, t)]. \quad (\text{A.6})$$

where

$$\begin{aligned} c(x) &= \lim_{\tau \rightarrow 0} \frac{1}{\tau} \int_{-\infty}^{\infty} q \kappa_\tau(q, x) dq, \quad \text{and} \\ d(x) &= \lim_{\tau \rightarrow 0} \frac{1}{2\tau} \int_{-\infty}^{\infty} q^2 \kappa_\tau(q, x) dq. \end{aligned} \quad (\text{A.7})$$

Coefficients of the space-use equation arising from a spatially-dependent resource selection model

The redistribution kernel for the spatially-explicit resource selection model (Eq.7) is given by

$$k(x, x', \tau) = \frac{\phi_\tau(x - x')w(x)}{\int \phi_\tau(x'' - x')w(x'')dx''} \quad (\text{A.8})$$

where $\phi_\tau(x - x')dx'dx$ the probability that an individual located between x' and $x' + dx$ will move to a location between x and $x + dx$ away from x , and $w(x)$ is a resource selection function. The denominator is a normalizing factor that ensures that Eq. (A.1) holds.

We consider the case of continuous, sufficiently smooth positive preference function $w(x)$, and a bounded symmetric distribution of displacement distances $\phi_\tau(q)$ (Figure 4). Recalling the definition (A.2), we can rewrite A.8, as

$$\kappa_\tau(q, x) = \frac{\phi_\tau(q)w(x + q)}{\int \phi_\tau(q')w(x + q')dq'} \quad (\text{A.9})$$

The p^{th} moment of the distribution of displacement distances is

$$M_p(\tau) := \int q^p \phi_\tau(q) dq. \quad (\text{A.10})$$

Since ϕ_τ is a probability density function $M_0(\tau) = 1$, and since it is symmetric $M_1(\tau) = 0$ and all higher odd moments are zero. $M_2(\tau)$ is the variance, and we make the conventional assumption that in the limit $\tau \rightarrow 0$ the higher even moments can be neglected (*i.e.*, they vanish faster than linearly in τ).

Under the above assumptions, Taylor expanding w yields the following expression for the denominator of (A.9),

$$\int \phi_\tau(q')w(x + q')dq' = w(x) + \frac{w_{xx}(x)}{2!}M_2(\tau) + \dots \quad (\text{A.11})$$

where $w_{xx} = \frac{d^2w}{dx^2}$. In the limit $\tau \rightarrow 0$, only the first term of Eq. A.11 remains. Using this expression for the denominator and inserting (A.9) into Eqs. (A.7) yields

$$\begin{aligned} c(x) &= \lim_{\tau \rightarrow 0} \frac{1}{\tau} \frac{w_x(x)M_2(\tau) + w_{xxx}(x)M_4(\tau)/3! + \dots}{w(x) + w_{xx}(x)M_2(\tau)/2! + \dots} \\ &= \lim_{\tau \rightarrow 0} \frac{M_2(\tau)}{\tau} \cdot \frac{w_x(x)}{w(x)} \end{aligned} \quad (\text{A.12})$$

where $w_x = \frac{dw}{dx}$, etc. Similarly,

$$\begin{aligned} d(x) &= \lim_{\tau \rightarrow 0} \frac{1}{2\tau} \frac{w(x)M_2(\tau) + w_{xx}(x)M_4(\tau)/2! \dots}{w(x) + w_{xx}(x)M_2(\tau)/2 + \dots} \\ &= \lim_{\tau \rightarrow 0} \frac{M_2(\tau)}{2\tau}. \end{aligned} \quad (\text{A.13})$$

This is Eq (12a,b).

As two examples, we consider two special cases of the distribution of displacement distances, (i) the fixed step length $\phi_\tau(q) = (\delta(q - L) + \delta(q + L))/2$ for which $\lim_{\tau \rightarrow 0} \frac{M_2(\tau)}{\tau} = 1$, and, (ii) the exponential step distribution (Figure 4), namely $\phi_\tau(q) = (1/2L)e^{-|q|/L}$, which gives $\lim_{\tau \rightarrow 0} \frac{M_2(\tau)}{\tau} = 2$. As is clear from (A.12) and (A.13), it is the value of the second moment of ϕ_τ that determines the magnitude of the advection and diffusion coefficients.

Steady-state distribution of the resource selection model

The steady state distribution u^* for Equation (9) can be derived by first expressing the equation in conservation law form

$$\frac{\partial u}{\partial t} + \frac{\partial}{\partial x} [(du)_x - cu] = 0. \quad (\text{A.14})$$

where the quantity in square brackets is the flux. Setting the time derivative to zero implies that the steady-state flux is a constant independent of spatial position. Since by definition the steady-state pattern of space-use u^* does not change with time, this constant is zero. Thus in the steady state $(du)_x = cu$, which may be integrated to give the steady state pattern of space use

$$u^*(x) = \frac{C}{d(x)} \exp \int^x \frac{c(x')}{d(x')} dx' \quad (\text{A.15})$$

with C chosen so that $u^*(x)$ integrates to 1. Substituting (A.12) and (A.13) into A.15 gives Eq. (13).

Note that if the individual is moving in a finite, bounded region, then zero-flux boundary conditions apply to A.6 at the edges of the domain, i.e.:

$$\frac{\partial}{\partial x} [(d(x)u(x, t)) - c(x)u(x, t)] = 0, \quad \text{at } \partial\Omega \quad (\text{A.16})$$

In this case a nonzero steady-state solution Eq. A.15 is guaranteed to exist for any continuous $w(x)$.

Literature Cited

Gardiner, C. W. 1983. Handbook of stochastic methods. First Edition. Springer, Berlin, Germany, p. 249.