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Appendix A. Numerical calculation of model parameters.

# Numerical calculation of model parameters

Here we describe methods of numerical calculation of the parameters given in the main text. As supplementary material we also provide the programming script (Rcode.txt) that we used for numerical calculations as well as simulations, with the software R (R Development Core Team, 2009). The approach used here is essentially a second degree Newton-Cotes quadrature, which works well for our purposes but is just one of many possible methods for numerical integration.

The main idea behind the numerical calculation is to discretize the model, by dividing the state space  $\Omega$  into a number of small intervals of length  $\Delta x$ . Let  $\mathbf{x}$  denote the vector of the discretized state space. The accuracy of parameter values will increase as the interval  $\Delta x$  is made smaller. Similarly, for the cases with non-constant environment let  $\mathbf{z}$  denote the vector of possible values for the environmental variable  $Z$ . The probability of choosing element  $z_i$  a given year is  $P(Z = z_i)$  (we have  $\sum_i P(Z = z_i) = 1$ ). Letting  $k(y, x, z_i)$  denote the projection function in environment  $z_i$ , the mean projection function across environments is given by  $k(y, x) = \sum_i k(y, x, z_i)P(Z = z_i)$ . The mean survival probability function across environments is given by  $\bar{s}(x) = \sum_i s(x, z_i)P(Z = z_i)$ , and similar mean functions can be found for all other vital parameters (fecundity  $b(x, z)$ , variance in fecundity  $\sigma_B^2(x, z)$ , the covariance  $\sigma_{BS}^2(x, z)$  the transition functions  $f_s(y; x, z)$  and  $f_b(y; x, z)$ ).

## Expected growth rate, stable distribution and reproductive value

The growth rate  $\lambda$ , stable distribution  $u(x)$  and reproductive value function  $v(x)$ , are found by discretizing the mean projection function  $k(y, x)$  to obtain a (large) projection matrix  $\mathbf{K}$  (Ellner and Rees, 2006). Then the discrete, deterministic model  $\mathbf{n}_{t+1} = \mathbf{K}\mathbf{n}_t$  can be iterated, where  $\mathbf{n}_t$  is the population vector giving the (expected) number of

individuals in each state interval at time  $t$ . The initial population vector can be arbitrarily chosen, but must contain at least one reproducing individual. After some time steps, when the stable distribution is reached, the growth rate is given by  $\lambda \approx \frac{N_{t+1}}{N_t}$ , where  $N_t$  is the total population size at time  $t$ , and the stable distribution vector is given by  $\mathbf{u} \approx \frac{\mathbf{n}_t}{N_t}$  (Caswell, 2001). To calculate the reproductive value function  $v(x)$ , we first iterate the transposed model  $\mathbf{n}_{t+1} = \mathbf{K}^T \mathbf{n}_t$ . The reproductive value vector  $\mathbf{v}$  is then the stable distribution of this transposed model (Caswell, 2001), scaled so that  $\mathbf{v}\mathbf{u} = 1$ . Based on interpolation of the data points in  $\mathbf{u}$  and  $\mathbf{v}$  and appropriate scaling, approximations for the functions  $u(x)$  and  $v(x)$  are obtained. Alternatively,  $\lambda$  can be found as the dominant eigenvalue of  $\mathbf{K}$ , and  $\mathbf{u}$  and  $\mathbf{v}$  are the corresponding right and left eigenvectors, with the same scaling as above (Caswell, 2001). Let  $\hat{\lambda}$ ,  $\hat{u}(x)$  and  $\hat{v}(x)$  denote the estimated values of  $\lambda$ ,  $u(x)$  and  $v(x)$ . Once these are found, they can be used further in the estimation of the demographic and environmental variance.

## Demographic variance

With no environmental stochasticity, the demographic variance is given by equation (5), and when both types of stochasticity are included it is given by equation (8). In latter case, we must use the mean vital parameter functions over environments such as  $\bar{s}(x)$  (see first paragraph). To estimate the demographic variance, we must first find the expectation and variance of the stochastic variables  $v(Y_{sx})$  and  $v(Y_{bx})$  (next year's reproductive value of an individual of state  $x$  and its offspring, respectively), for each element of the state vector. For an element  $x_i$  the expectation of  $v(Y_{sx_i})$  is found as  $\hat{\mu}_{vs_i} = \sum_j \bar{f}_s(x_j; x_i) \hat{v}(x_j) \Delta x$ . Letting  $\hat{\mu}_{vs_i}^* = \sum_j \bar{f}_s(x_j; x_i) \hat{v}^2(x_j) \Delta x$ , the variance is given by  $\hat{\sigma}_{vs_i}^2 = \hat{\mu}_{vs_i}^* - \hat{\mu}_{vs_i}^2$ . Similar estimates are found for the mean and variance of  $v(Y_{bx})$ . Finally, the estimate of the demographic variance is found by summing up all the elements entering the formula,

$$\hat{\sigma}_d^2 = \sum_j \hat{u}(x_j) \left[ \bar{s}(x_j) \hat{\sigma}_{vs_i}^2 + \bar{b}(x_j) \hat{\sigma}_{vb_i}^2 + \hat{\mu}_{vs_j}^2 \bar{s}(x_j) (1 - \bar{s}(x_j)) + \hat{\mu}_{vb_j}^2 \bar{\sigma}_B^2(x_j) + 2\hat{\mu}_{vs_j} \hat{\mu}_{vb_j} \bar{\sigma}_{BS}^2(x_j) \right] \Delta x.$$

For the simpler case of a constant environment, replace  $\bar{s}(x_j)$  by  $s(x_j)$  etc. in the above formula.

### Environmental variance

To calculate the environmental variance (equations 7 and 9), we first show that this parameter is approximately equal to the variance of the growth rate  $\lambda(Z)$  with respect to the environmental variable  $Z$ . Let  $k(y, x)$  be the mean projection function across environments as before, associated with the overall mean growth rate  $\lambda$ , the stable distribution  $u(x)$  and the reproductive value  $v(x)$ . A first order Taylor approximation of  $\lambda(Z)$  around  $\lambda$  gives

$$\begin{aligned} \lambda(Z) &\approx \lambda + \int \int \frac{\partial \lambda}{\partial k(y, x)} [k(y, x, Z) - k(y, x)] dy dx \\ &= \int \int v(y) u(x) k(y, x, Z) dy dx \\ &= \int u(x) E[W_x | Z] dx. \end{aligned}$$

This approximation works well as long as the environment shows small fluctuations, so that effects of the environment on the stable distribution and reproductive value function can be ignored. By using the Taylor approximation, the variance of  $\lambda(Z)$  with respect to

environment is given by

$$\begin{aligned}\text{Var}(\lambda(Z)) &\approx \text{Var}\left(\int u(x)\text{E}[W_x|Z]dx\right) \\ &= \int \int u(x)u(y)c(x,y)dydx,\end{aligned}$$

where  $c(x,y) = \text{Cov}(\text{E}[W_x|Z], \text{E}[W_y|Z])$ . Thus, we see that the variance of  $\lambda(Z)$  corresponds approximately to the environmental variance derived in the main text (equation 8).

To find this variance numerically, we first find the growth rates  $\lambda(z_i)$  for each value of the environmental vector, using the methods given above for each projection function  $k(y, x, z_i)$ . The numerical approximation of the environmental variance is then given by

$$\hat{\sigma}_e^2 \approx \sum_i (\lambda(z_i) - \hat{\lambda})^2 P(Z = z_i).$$

## Literature cited

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