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Appendix C: Symmetric predation and the existence of alternative states

We impose the following symmetry assumptions:

$$K_1 = K_2 = K\,,\; e_1 = e_2 = e\,,\; \alpha_1 = \alpha_2 = \alpha\,,\; \tau_1 = \tau_2 = \tau\,. \tag{C.1}$$

As above, and without loss of generality, we assume $\theta_2 = -\theta_1 = \theta > 0$.

In this symmetric case, for which most of the main findings have been presented, simpler and more detailed results can be deduced. As in the general case (Appendix B), we derive the univariate transcendental equation from which the equilibrium trait values $\hat{\bar{x}}$ for the coexistence equilibria are obtained. Substantial simplifications are obtained because the admissible equilibrium values $\hat{\bar{x}}$ depend only on three compound parameters. In addition, we are able to provide sufficient conditions as well as necessary conditions for the (simultaneous) existence of one, three, or five coexistence equilibria. In this appendix, we focus on coexistence equilibria.

We introduce the compound parameters

$$\gamma = \frac{Ke\alpha\tau}{d\sqrt{\sigma^2+\tau^2}}\,, \quad O = \frac{\theta}{\sqrt{\sigma^2+\tau^2}}\,, \tag{C.2a}$$

and the scaled trait variable

$$z = \frac{\bar{x}}{\sqrt{\sigma^2+\tau^2}}\,. \tag{C.2b}$$

Thus, without loss of generality, we are measuring the quantitative trait x in units of $\sqrt{\sigma^2+\tau^2}$. Note that this differs from the general case in Appendix B.

If we write

$$X_1 = \exp\left[\frac{1}{2}(z+O)^2\right]\,, \quad X_2 = \exp\left[\frac{1}{2}(z-O)^2\right]\,, \tag{C.3}$$

the equilibrium abundances become

$$\hat{N}_1 = KX_1\,\frac{(r_2/\gamma)X_2^2 - r_2X_2 + r_1X_1}{r_1X_1^2 + r_2X_2^2}\,, \tag{C.4a}$$

$$\hat{N}_2 = KX_2\,\frac{(r_1/\gamma)X_1^2 - r_1X_1 + r_2X_2}{r_1X_1^2 + r_2X_2^2}\,, \tag{C.4b}$$

$$\hat{P} = \frac{r_1r_2\sqrt{\sigma^2+\tau^2}}{\alpha\gamma\tau}X_1X_2\frac{\gamma(X_1+X_2) - X_1X_2}{r_1X_1^2 + r_2X_2^2}\,. \tag{C.4c}$$

If these are substituted into the differential equation (1c), one obtains the equilibrium values $\hat{z}$ as the zeros of the transcendental function

$$\phi(z) = \rho g(z) - g(-z)\,, \tag{C.5}$$

where

$$\rho = \rho_1/\rho_2 \tag{C.6}$$

and

$$g(z) = X_1^2(O - z) - 2\gamma O X_1\,. \tag{C.7}$$

Then the equilibrium solutions $\hat{\bar{x}}$ are obtained by solving $\phi(z) = 0$, which depends only on the *three* parameters γ, O, and ρ. Notice that

$$\phi(z) = \frac{\sqrt{\sigma^2 + \tau^2}}{d}(\rho X_1^2 + X_2^2)\frac{d\overline{W}}{d\bar{x}}(\bar{x})\,. \tag{C.8}$$

The supersymmetric case, $\rho = 1$

Here, we study the properties of $\phi(z)$ if $\rho = 1$, in particular, the number of possible solutions of $\phi(z) = 0$ $(-O < z < O)$. Because $\phi(z) = g(z) - g(-z)$ is an odd function, $z = 0$ is always a solution, and the number of solutions is odd. The observation

$$\phi(-O) = -\phi(O) = 2O[1 + \gamma(e^{2O^2} - 1)] > 2O > 0 \tag{C.9}$$

will play an important role.

To derive results about the number of possible solutions, we compute the derivative of ϕ with respect to z. We obtain

$$\phi'(z) = (X_1^2 + X_2^2)(2O^2 - 2z^2 - 1) - 2\gamma O[X_1(O + z) + X_2(O - z)]\,, \tag{C.10}$$

hence

$$\phi'(0) = 4e^{\frac{1}{2}O^2}\left[e^{\frac{1}{2}O^2}(O^2 - \frac{1}{2}) - \gamma O^2\right]. \tag{C.11}$$

It follows that $\phi'(z) \leq 0$ for every $z \in [-O, O]$ if

$$O^2 \leq \frac{1}{2}\,, \tag{C.12}$$

whence we conclude that $z = 0$ is the only solution if (C.12) holds. In addition, it follows from (C.10) that for arbitrary given $O > 0$, $z = 0$ is the only solution of $\phi(z) = 0$ if γ is sufficiently large.

Next, we observe that $\phi'(0) > 0$ if and only if

$$O^2 > \frac{1}{2} \tag{C.13}$$

and

$$\gamma < \gamma_1 = \frac{2O^2 - 1}{2O^2}\, e^{\frac{1}{2}O^2}\,. \tag{C.14}$$

In view of (C.9), we infer that there exist at least three solutions if (C.13) and (C.14) are satisfied.

Finally, we show that five solutions of $\phi(z) = 0$ may exist. To this aim, we develop ϕ into a series about $z = 0$:

$$\phi(z) = \phi'(0)z + \frac{2e^{\frac{1}{2}O^2}}{3}\left[e^{\frac{1}{2}O^2}(4O^4 - 3) - \gamma O^2(O^2 + 3)\right] z^3 + O(z^5)\,. \tag{C.15}$$

From this, together with (C.9), we conclude that five solutions exist if $\phi'(0) < 0$ and the term in brackets is positive. These two conditions are satisfied if and only if

$$\gamma_1 < \gamma < \gamma_2\,, \tag{C.16}$$

where

$$\gamma_2 = \frac{4O^4 - 3}{O^2(O^2 + 3)}\, e^{\frac{1}{2}O^2}\,. \tag{C.17}$$

(C.16) can only be satisfied if $\gamma_1 < \gamma_2$. A simple calculation shows that this is the case if and only if

$$6O^4 - 5O^2 - 3 > 0\,, \tag{C.18}$$

which holds if and only if

$$O = O_c > \sqrt{\frac{5 + \sqrt{97}}{12}} \approx 1.11239\,. \tag{C.19}$$

We conclude that there exist five solutions of $\phi(z) = 0$ if (C.19) holds and γ is slightly larger than γ_1. If γ increases, the two solutions bifurcating off $z = 0$ get larger in absolute value and hit the 'outer' solutions (near $\pm O$) at a critical value γ_c which is (at least usually) less than

γ_2. If $\gamma > \gamma_c$, (apparently) only the solution $z = 0$ exists. This is in accordance with our above finding that for sufficiently large γ, $z = 0$ is the only solution.

Remark 1. In terms of the original parameters, the conditions for obtaining 1, 3, or 5 solutions are not always immediately intuitive or simple. For instance, condition (C.12) becomes $\sigma^2 + \tau^2 \geq 2\theta^2$. Although this is simple, it is sharp only if $Ke\alpha/d$ is small. Alternatively, condition (C.19) translates into the simple inequality $\theta^2 > O_c^2(\sigma^2 + \tau^2)$. However, condition $\gamma > \gamma_1$ from (C.16) translates into

$$\frac{Ke\alpha}{d} > \frac{2\theta^2(\sigma^2 + \tau^2) - 1}{\tau\sqrt{\sigma^2 + \tau^2}} \exp[-\tfrac{1}{2}\theta^2(\sigma^2 + \tau^2)]\,, \tag{C.20}$$

which does not give a simple condition for σ^2. Nevertheless, the right-hand side is strictly monotone increasing in σ^2 and thus yields an (implicit) *upper* bound for σ^2.