

Appendix C. The derivative of a GP, $f'(A)$.

C-1: Construction of the joint distribution of a GP, $f(A)$, and its derivative, $f'(A)$

Given a GP, i.e., $f(A) \sim \text{GP}[\mu(A), \Sigma_{f,f}(A, A')]$, its derivative, $f'(A)$, is also a GP with the mean function $\mu'(A)$ and the covariance function $\Sigma_{f',f'}(A, A')$ (Rasmussen and Williams 2006).

Specifically,

$$(C.1) \quad \frac{df}{dA} \equiv f'(A) \sim \text{GP}[\mu'(A), \Sigma_{f',f'}(A, A')].$$

where $\mu'(A)$ is given by

$$(C.2) \quad \mu'(A) = \text{E}[\partial f(A)/\partial A] = (\partial / \partial A)\text{E}[f(A)] = \beta / \max(A),$$

and $\Sigma_{f',f'}(A, A')$ is given by

$$(C.3) \quad \begin{aligned} \Sigma_{f',f'}(A, A') &= \text{E} \left[\left[\frac{\partial f(A)}{\partial A} - \frac{\partial \mu(A)}{\partial A} \right] \left[\frac{\partial f(A')}{\partial A'} - \frac{\partial \mu(A')}{\partial A'} \right]^T \right] \\ &= \frac{\partial}{\partial A} \text{E}[f(A) - \mu(A)][f(A') - \mu(A')]^T \frac{\partial}{\partial A'} \\ &= \frac{\partial^2}{\partial A \partial A'} \Sigma_{f,f}(A, A') \\ &= 2\phi\tau^2 \left[1 - 2\phi \left| \frac{A - A'}{\max(A)} \right|^2 \right] \exp \left(-\phi \left| \frac{A - A'}{\max(A)} \right|^2 \right). \end{aligned}$$

Similarly, we can determine the covariance between $f(A)$ and $f'(A)$, i.e.,

$$(C.4) \quad \begin{aligned} \Sigma_{f,f'}(A, A') &= \text{E} \left\{ [f(A) - \mu(A)] \left[\frac{\partial f(A')}{\partial A'} - \frac{\partial \mu(A')}{\partial A'} \right]^T \right\} \\ &= \text{E} \left\{ [f(A) - \mu(A)][f(A') - \mu(A')]^T \left(\frac{\partial}{\partial A'} \right)^T \right\} \\ &= \frac{\partial}{\partial A} \Sigma_{f,f}(A, A') \end{aligned}$$

$$= 2\phi\tau^2 \left[\frac{A - A'}{\max(A)} \right] \exp \left[-\phi \left| \frac{A - A'}{\max(A)} \right|^2 \right].$$

As described in the Methods, we need to construct the joint distribution of $f(A)$ and $f'(A)$ evaluated at a specific value of A . Because $f'(A)$ is a linear functional of $f(A)$, their joint distribution is also normal. Putting everything together, the bivariate normal distribution is given by:

$$(C.5) \quad \begin{pmatrix} f(A) \\ f'(A) \end{pmatrix} \sim N \left[\begin{pmatrix} \ln\alpha + \beta A/\max(A) \\ \beta/\max(A) \end{pmatrix}, \begin{pmatrix} \Sigma_{f,f}(A, A) & \Sigma_{f,f'}(A, A) \\ \Sigma_{f,f'}(A, A)^T & \Sigma_{f',f'}(A, A) \end{pmatrix} \right].$$

where $\Sigma_{f,f}(A, A)$ is the variance in the GP evaluated at A , $\Sigma_{f',f'}(A, A)$ is the variance of $f'(A)$ at A , and $\Sigma_{f,f'}(A, A)$ is the covariance between $f(A)$ and $f'(A)$ evaluated at A . They can be obtained by plugging $A = A'$ into Eqs 6, C.3, and C.4:

$$(C.6) \quad \Sigma_{f,f}(A, A) = \tau^2$$

$$(C.7) \quad \Sigma_{f,f'}(A, A) = 0$$

$$(C.8) \quad \Sigma_{f',f'}(A, A) = 2\phi\tau^2.$$

C-2: Calculation of the benchmark probability

As described in Method section (section 2), π is the posterior probability of the presence of Allee effects. A good benchmark probability against which to compare π for detecting Allee effects is the probability of the presence of Allee effects prior to incorporating data, $\Pr[f(0) \leq 0, f'(0) > 0]$. Thinking about $f(0)$ and $f'(0)$ as the x and y axes, we are essentially trying to find the probability of being in the second quadrant. To obtain this probability, we first consider the conditional probability, $\Pr[f(0), f'(0)|\theta]$, where

$$(C.9) \quad f(0), f'(0) | \boldsymbol{\theta} \sim N \left[\begin{pmatrix} \ln \alpha \\ \beta / \max(A) \end{pmatrix}, \begin{pmatrix} \tau^2 & 0 \\ 0 & 2\phi\tau^2 \end{pmatrix} \right],$$

After marginalizing over the independent, zero-mean priors for $\ln \alpha$ and β (see Appendix B-1), the distribution for $\{f(0), f'(0)\}$ is centered on $(0,0)$. Moreover, since the covariance is 0, this joint distribution is symmetric about the horizontal and vertical axes. Hence, each quadrant has equal probability. Therefore, prior to incorporating data, our index of Allee effect presence is precisely 1/4. It is important to note that as we update the distribution for $\boldsymbol{\theta}$, the symmetry of the marginal distribution for $\{f(0), f'(0)\}$ is broken and any value for π in $(0,1)$ may emerge.

LITERATURE CITED

Rasmussen, C. E., and C. K. I. Williams. 2006. Gaussian Processes for Machine Learning. The MIT Press.