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Scaling from trees to forests: tractable macroscopic equations for forest dynamics.
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Appendix E. Flat-top model: Stability of the single-species stationary state given by (24-26) in the text.

Appendix B shows that, if $\eta(D,t)$ and $\varepsilon(t)$ are small perturbations of the stationary distribution (24) and equilibrium canopy-tree diameter respectively, then $\varepsilon(t)$ is of the same order as $\eta(D,t)$. Thus, a small amount of cutting or planting means a correspondingly small change in D^*. Also, it is easy to show that $\frac{d\varepsilon(t)}{dt}$ is of order $\varepsilon(t)$ or smaller. This means that we can chose the initial perturbation to be small enough so that dD^*/dt immediately after perturbation is of smaller order than G_D and G_L, and this, in turn, ensures that each cohort after perturbation will cross into the canopy only once (unlike the second oldest cohort in Figure 17b).

Now imagine that we perturb an equilibrium monoculture by cutting a few canopy trees. This causes the canopy height to decrease instantaneously $\hat{D}^* > D^*(t_0) = \hat{D}^* + \varepsilon(t_0)$. Although we consider the case in which the perturbation $\varepsilon(t_0)$ is negative and in which only canopy trees are perturbed, similar arguments for the other cases lead to the same stability criterion. Now, let $\gamma(\tau)$ be the fraction of trees cut in the cohort that crossed into the canopy τ years before the perturbation. We must now consider three classes of cohorts:

1. Cohorts in the canopy, both before and after the perturbation.
2. Cohorts which crossed into the canopy at the moment of the perturbation.
3. Cohorts in the understory before and immediately after the perturbation or cohorts not born until after the perturbation.

Then the equation for canopy height at time t is:

$$\begin{aligned} 1 = &\int_0^\infty (1-\gamma(\tau)) F e^{-\theta_1 \hat{D}^*} e^{-\mu_L \tau + t} \alpha(\hat{D}^* + G_L(\tau + t))^2 d\tau \\ &+ \int_{\hat{D}^*+\varepsilon(t_0)}^{\hat{D}^*} F e^{-\theta_1 (D)} e^{-\mu_L t} \alpha(D + G_L t)^2 dD / G_L \\ &+ \int_0^t F e^{-\theta_1 (\hat{D}^* + \varepsilon(t-\tau))} e^{-\mu_L \tau} \alpha(\hat{D}^* + \varepsilon(t-\tau) + G_L \tau)^2 d\tau, \end{aligned} \tag{E-1}$$

where $\theta_1 = \frac{\mu_D}{G_D}$.

The first term in (E-1) governs cohorts in class one. It gives the sum at time t of the uncut fraction $(1-\gamma(t))$ of crown areas of all cohorts that had crossed into the canopy τ years before the perturbation (i.e. they have spent $t+\tau$ years in the canopy). The second term describes class 2. Here, to make the calculations easier, we integrate over the initial sizes of the cohorts, rather than their ages. The third term governs class 3 cohorts, which have spent less than t years in the canopy. The variable of integration (τ) is the amount of time that the cohort has been in the canopy at time t.

To derive a relation between $\varepsilon(t_0)$ and $\gamma(\tau)$, we note that at the moment of perturbation we have:

$$\int_0^\infty \gamma(\tau) F e^{-\theta_1 \hat{D}^*} e^{-\mu_L \tau} \alpha(\hat{D}^* + G_L \tau)^2 d\tau = \int_{\hat{D}^*+\varepsilon(t_0)}^{\hat{D}^*} F e^{-\theta_1 D} \alpha D^2 dD. \tag{E-2}$$

The left-hand integral is the total area lost because of the perturbation and it is balanced on the right-hand side by the area gained by the resulting drop in D^*. Since the initial perturbation $\varepsilon(t_0)$ is small we can linearize the integral in the right hand side of (E-1) to get

$$\int_{\hat{D}^*+\varepsilon(t_0)}^{\hat{D}^*} F e^{-\theta_1 D} \alpha D^2 dD \approx -\varepsilon(t_0) F e^{-\theta_1 \hat{D}^*} \alpha \hat{D}^{*2}. \tag{E-3}$$

We now substitute this result into (E-2) and rearrange:

$$\varepsilon(t_0) \approx -\frac{\int_0^\infty \gamma(\tau) F e^{-\theta_1 \hat{D}^*} e^{-\mu_L \tau} \alpha(\hat{D}^* + G_L \tau)^2 d\tau}{F e^{-\theta_1 \hat{D}^*} \alpha \hat{D}^{*2}} = -\int_0^\infty \gamma(\tau) e^{-\mu_L \tau} \left(1 + \frac{G_L}{\hat{D}^*}\tau\right)^2 d\tau. \tag{E-4}$$

Our objective is to derive dynamics of the perturbation $\varepsilon(t)$, if this is small, and so we linearize (E-1) by ignoring terms of order $\varepsilon^2(t)$ or smaller. After linearizing and using (E-3), (E-1) becomes:

$$\begin{aligned} 1 \approx &\int_0^\infty F e^{-\theta_1 \hat{D}^*} e^{-\mu_L (\tau+t)} \alpha(\hat{D}^* + G_L(\tau+t))^2 d\tau \\ &- \int_0^\infty \gamma(\tau) F e^{-\theta_1 \hat{D}^*} e^{-\mu_L (\tau+t)} \alpha(\hat{D}^* + G_L(\tau+t))^2 d\tau \\ &- e^{-\mu_L t} F \alpha \varepsilon(t_0) e^{-\theta_1 \hat{D}^*} \alpha(\hat{D}^* + G_L t)^2 \\ &+ \int_0^t F e^{-\theta_1 \hat{D}^*} e^{-\mu_L \tau} \alpha(\hat{D}^* + G_L \tau)^2 d\tau \\ &- \int_0^t \varepsilon(t-\tau) \theta_1 F e^{-\theta_1 \hat{D}^*} e^{-\mu_L \tau} \alpha(\hat{D}^* + G_L \tau)^2 d\tau \\ &+ \int_0^t \varepsilon(t-\tau) F e^{-\theta_1 \hat{D}^*} e^{-\mu_L \tau} \alpha 2(\hat{D}^* + G_L \tau) d\tau. \end{aligned} \tag{E-5}$$

The sum of the first and the forth terms is one because it equals the right hand side of the PPA equation at the stationary state (71), and so we have:

$$
\begin{aligned}
0 \approx & -\int_0^\infty \gamma(\tau) F e^{-\theta_1 \hat{D}^*} e^{-\mu_L(\tau+t)} \alpha(\hat{D}^* + G_L(\tau+t))^2 d\tau \\
& - e^{-\mu_L t} F \alpha \varepsilon(t_0) e^{-\theta_1 \hat{D}^*} \alpha(\hat{D}^* + G_L t)^2 \\
& - \int_0^t \varepsilon(t-\tau)\theta_1 F e^{-\theta_1 \hat{D}^*} e^{-\mu_L \tau} \alpha(\hat{D}^* + G_L \tau)^2 d\tau \\
& + \int_0^t \varepsilon(t-\tau) F e^{-\theta_1 \hat{D}^*} e^{-\mu_L \tau} \alpha 2(\hat{D}^* + G_L \tau) d\tau.
\end{aligned} \tag{E-6}
$$

Like the second term, the first term is of the form $e^{-\mu_L t} Q(t^2)$ where $Q(t^2)$ is a quadratic function of time because:

(E-7)

$$
\begin{aligned}
\int_0^\infty & \gamma(\tau) F e^{-\theta_1 \hat{D}^*} e^{-\mu_L(\tau+t)} \alpha(\hat{D}^* + G_L(\tau+t))^2 d\tau = \\
& = e^{-\mu_L t} \int_0^\infty \gamma(\tau) F e^{-\theta_1 \hat{D}^*} e^{-\mu_L \tau} \alpha(\hat{D}^* + G_L \tau)^2 d\tau \\
& + e^{-\mu_L t} 2 t G_L \int_0^\infty \gamma(\tau) F e^{-\theta_1 \hat{D}^*} e^{-\mu_L \tau} \alpha(\hat{D}^* + G_L \tau) d\tau \\
& + e^{-\mu_L t} t^2 \int_0^\infty \gamma(\tau) F e^{-\theta_1 \hat{D}^*} e^{-\mu_L \tau} \alpha G_L^2 d\tau.
\end{aligned}
$$

Thus, (E-6) becomes:

$$
\begin{aligned}
0 \approx & -\int_0^t \varepsilon(t-\tau)\theta_1 e^{-\mu_L \tau} (\hat{D}^* + G_L \tau)^2 d\tau + \\
& \int_0^t \varepsilon(t-\tau) e^{-\mu_L \tau} 2(\hat{D}^* + G_L \tau) d\tau + e^{-\mu_L t} Q(t^2).
\end{aligned} \tag{E-8}
$$

We now substitute a new variable $\zeta = t - \tau$ into (E-8), divide through by $e^{-\mu_L t}$

$$
\begin{aligned}
0 \approx \theta_1 & \int_0^t \varepsilon(\zeta) e^{\mu_L \zeta} (\hat{D}^* + G_L(t-\zeta))^2 d\zeta - \\
& \int_0^t \varepsilon(\zeta) e^{\mu_L \zeta} 2(\hat{D}^* + G_L(t-\zeta)) d\zeta + Q(t^2),
\end{aligned} \tag{E-9}
$$

and differentiate three times with respect to t. Then we get the second order homogeneous differential equation with respect to $\varepsilon(t)$

$$
\begin{aligned}
0 \approx & \varepsilon''(t) \left(\frac{\mu_D}{G_D} (\hat{D}^*)^2 - 2\hat{D}^* \right) + \\
& \varepsilon'(t) \left(\mu_L \theta_1 (\hat{D}^*)^2 - 4\mu_L \hat{D}^* + 4 G_L \theta_1 \hat{D}^* - 2G_L \right) + \\
& \varepsilon(t) \left(4G_L^2 \theta_1 + 2\mu_L^2 \frac{\mu_D}{G_D} (\hat{D}^*)^2 + 4 G_L \mu_L \frac{\mu_D}{G_D} \hat{D}^* - \mu_L^2 2\hat{D}^* - 2G_L \mu_L \right).
\end{aligned} \tag{E-10}
$$

To find the general solution of this equation we substitute $\varepsilon(t) = e^{\lambda t}$ and after algebraic transformations we get the following eigenvalue equation:

$$
0 = \hat{D}^*(2-\theta_1 \hat{D}^*)(\lambda+\mu_L)^2 + 2G_L(1-\theta_1 \hat{D}^*)(\lambda+\mu_L) - 2\theta_1 G_L^2. \tag{E-11}
$$

We note that a simple shortcut to (E-11) is to substitute $\varepsilon(t) = e^{\lambda t}$ directly into equation (E-9).

The stationary state $\hat{D}^*$ is stable when $Re[\lambda] < 0$. To find a domain in the parameter manifold where $\hat{D}^*$ is stable let us denote $\Lambda = (\lambda + \mu_L)$, then for the real part of Λ we have

$$
\begin{aligned}
& \mathrm{Re}[\Lambda_{1,2}] = \\
& \mathrm{Re}\left[\frac{-G_L(1-\theta_1 \hat{D}^*) \pm G_L \sqrt{(1-\theta_1 \hat{D}^*)^2 + 2\hat{D}^*(2-\theta_1 \hat{D}^*)\theta_1}}{\hat{D}^*(2-\theta_1 \hat{D}^*)} \right]
\end{aligned} \tag{E-12}
$$

From (E-12) we have the following intervals for Λ:

1. $1 + \sqrt{2} < \hat{D}^* \theta_1$ - two complex eigenvalues with negative real parts.
2. $2 < \hat{D}^* \theta_1 \leqslant 1 + \sqrt{2}$ - two negative real eigenvalues.
3. $1 \leqslant \hat{D}^* \theta_1 < 2$ - two real eigenvalues, one positive and one negative.
4. Also, from (E-11) we have that if $\hat{D}^* \theta_1 = 2$ there is one negative real eigenvalue.

These results for groups 1-3 (but not group 4) hold for λ as long as μ_L is small (so that $\lambda \approx \Lambda$). Also, if $\hat{D}^* \theta_1$ is less than or equal to one, then inequality (23) is violated and so the species cannot invade an empty habitat (to prove this, see equation (F-1).

The quantity $\hat{D}^* \theta_1$, is simply the natural logarithm of the expected lifetime reproductive success of a tree that has just made it to the canopy at equilibrium (see eg. 24 and 26 in the main text). Thus, if this quantity is greater than or equal to two, then the stationary distribution (24-26) in the text is locally stable.

When $1 \leqslant \hat{D}^* \theta_1 < 2$ the discriminant in (E-12) is positive and so there is one positive and one negative Λ . If the absolute value of Λ is larger than μ_L then a positive eigenvalue λ exists, and, so then the stationary state is unstable. However, it is easy to see that when $1 \leqslant \hat{D}^* \theta_1 < 2$:

(E-13)

$$
\lim_{\hat{D}^* \theta_1 \to 2-} \frac{-G_L(1-\theta_1 \hat{D}^*) + G_L \sqrt{(1-\theta_1 \hat{D}^*)^2 + 2\hat{D}^*(2-\theta_1 \hat{D}^*)\theta_1}}{\hat{D}^*(2-\theta_1 \hat{D}^*)} = +\infty.
$$

Therefore, if $\hat{D}^* \theta_1$ is close enough to 2 then the subtraction of an equally small μ_L will not cause the largest λ to be negative. If this quantity is less than 2 but greater than one, then the stationary state is usually unstable and simulations show monotonic growth of $\hat{D}^*$ followed by a rapid crash and then a cycle of repeated episodes of growth and crash. This oscillation is sufficiently exotic to require a separate treatment.