Ecological Archives M080-008-A1

Joseph O. Ogutu, Hans-Peter Piepho, Robin S. Reid, Michael E. Rainy, Russell L. Kruska, Jeffrey S. Worden, Meshack Nyabenge, and N. Thompson Hobbs. 2010. Large herbivore responses to water and settlements in savannas. *Ecological Monographs* 80:241–266.

Appendix A. The bivariate Gaussian log-linear model.

The loglinear response surface may be related to the bivariate normal density given by (Johnson et al. 2000, p. 251):

$$p(x, y, \phi_x, \phi_y, \sigma_x, \sigma_y, \rho) = \frac{1}{2\pi\sqrt{1-\rho^2}} \exp\left(-\frac{1}{2(1-\rho^2)} \left\{ \left(\frac{x-\phi_x}{\sigma_x}\right)^2 - 2\rho\left(\frac{x-\phi_x}{\sigma_x}\right)\left(\frac{y-\phi_y}{\sigma_y}\right) + \left(\frac{y-\phi_y}{\sigma_y}\right)^2 \right\} \right)$$

The exponent determines the shape of the bivariate normal density. Since we are modelling abundance on the log-scale, we need to compare the coefficients in the exponent with the regression parameters of model (1). Rearranging the exponent yields:

$$c\left(\frac{1}{\sigma_x^2} x^2 - \frac{2\phi_x}{\sigma_x^2} x + \frac{\phi_x^2}{\sigma_x^2} - \frac{2\rho}{\sigma_x \sigma_y} xy + \frac{2\rho\phi_x}{\sigma_x \sigma_y} y + \frac{2\rho\phi_y}{\sigma_x \sigma_y} x - \frac{2\rho\phi_x\phi_y}{\sigma_x \sigma_y} + \frac{1}{\sigma_y^2} y^2 - \frac{2\phi_y}{\sigma_y^2} y + \frac{\phi_y^2}{\sigma_y^2} \right) \qquad (2)$$

with

$$c = -\frac{1}{2(1-\rho^2)}$$

Comparing coefficients we find

$$\beta_1 = c\left(\frac{2\rho\phi_y}{\sigma_x\sigma_y} - \frac{2\phi_x}{\sigma_x^2}\right)$$

$$\beta_2 = c\frac{1}{\sigma_x^2}$$

$$\gamma_1 = c\left(\frac{2\rho\phi_x}{\sigma_x\sigma_y} - \frac{2\phi_y}{\sigma_y^2}\right)$$

$$\gamma_2 = c\frac{1}{\sigma_y^2}$$

$$\delta = -c\frac{2\rho}{\sigma_x\sigma_y}$$

This system of equations needs to be solved for the parameters of the bivariate normal density:

(1) Using

$$\sqrt{\beta_2\gamma_2} = -\frac{c}{\sigma_x\sigma_y}$$

and

$$\frac{\delta}{\sqrt{\beta_2\gamma_2}} = 2\rho$$

we find

$$\rho = \frac{\delta}{2\sqrt{\beta_2\gamma_2}}$$

(2) $c = -\dfrac{1}{2(1-\rho^2)} = -\dfrac{1}{2\left(1-\dfrac{\delta^2}{4\beta_2\gamma_2}\right)}$

(3) $\dfrac{c}{\beta_2} = \sigma_x^2 = -\dfrac{1}{2\beta_2\left(1-\dfrac{\delta^2}{4\beta_2\gamma_2}\right)}$

By analogy

$$\sigma_y^2 = -\frac{1}{2\gamma_2\left(1-\frac{\delta^2}{4\beta_2\gamma_2}\right)}$$

(4) $\frac{\beta_1}{\delta} = -\phi_y + \frac{\phi_x\sigma_y}{\sigma_x\rho}$

$$\frac{\gamma_1}{2\gamma_2} = \frac{\rho\phi_x\sigma_y}{\sigma_x} - \phi_y$$

$$\frac{\beta_1}{\delta} - \frac{\gamma_1}{2\gamma_2} = \frac{\phi_x\sigma_y}{\sigma_x\rho} - \frac{\rho\phi_x\sigma_y}{\sigma_x} = \phi_x\frac{\sigma_y\left(1-\rho^2\right)}{\sigma_x\rho}$$

$$\frac{\frac{\beta_1}{\delta}-\frac{\gamma_1}{2\gamma_2}}{\beta_2} = \phi_x\frac{\sigma_y\sigma_x\left(1-\rho^2\right)}{c\rho} = \phi_x\frac{1}{c\delta}$$

$$\phi_x = c\delta\times\frac{\frac{\beta_1}{\delta}-\frac{\gamma_1}{2\gamma_2}}{\beta_2} = c\times\left(\frac{\beta_1}{\beta_2}-\frac{\delta\gamma_1}{2\beta_2\gamma_2}\right) = \frac{\frac{\delta\gamma_1}{2\beta_2\gamma_2}-\frac{\beta_1}{\beta_2}}{2\left(1-\frac{\delta^2}{4\beta_2\gamma_2}\right)} = \frac{\frac{\delta\gamma_1}{4\beta_2\gamma_2}-\frac{\beta_1}{2\beta_2}}{1-\frac{\delta^2}{4\beta_2\gamma_2}}$$

whence

$$\phi_x = \frac{\delta\gamma_1 - 2\beta_1\gamma_2}{4\beta_2\gamma_2 - \delta^2}$$

By analogy we find

$$\phi_y = \frac{\delta\beta_1 - 2\gamma_1\beta_2}{4\beta_2\gamma_2 - \delta^2}$$

LITERATURE CITED

Johnson, N. L., S, Kotz, and N. Balakrishnan. 2000. Continuous multivariate distributions. Volume 1: Models and applications. Wiley, New York, New York, USA.