B   Solving a linear system in order to predict mass loss from a decomposition network.

Linear problems such as (12) are often solved by assuming a solution of the form $\boldsymbol{u}e^{-\lambda t}$ and plugging it into (12), resulting in a "characteristic equation" for the eigenvalues λ. The rates are then identified by substituting those values of k back into (12) to identify the eigenvectors $\boldsymbol{u}$. This approach was shown in equations (A.1) - (A.11).

We use a similar eigenvalue analysis to solve (12). First, we decompose $\mathbf{A}$ into its eigenvalues λ_i and eigenvectors $\boldsymbol{u_i}$

$$\mathbf{A} = \boldsymbol{U\Lambda U}^{-1} \tag{B.1}$$

where

$$\boldsymbol{\Lambda} = \begin{bmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \lambda_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \lambda_z \end{bmatrix} \tag{B.2}$$

$$\boldsymbol{U} = \begin{bmatrix} \boldsymbol{u}_1 & \boldsymbol{u}_2 & \cdots & \boldsymbol{u}_z \end{bmatrix} \tag{B.3}$$

Substituting (B.1) into (12), we get

$$\frac{d\mathbf{x}}{dt} = \mathbf{U\Lambda U}^{-1}\mathbf{x} + J(t)\mathbf{p_0} \tag{B.4}$$

Next, we change the state coordinate system x_i to one aligned with the eigenvectors. The system then becomes diagonal or parallel, in eigenspace. This is done by multiplying both sides of (12) by $\mathbf{U}^{-1}$,

$$\mathbf{U}^{-1}\frac{d\mathbf{x}}{dt} = \mathbf{\Lambda U}^{-1}\mathbf{x} + J(t)\mathbf{U}^{-1}\mathbf{p_0} \tag{B.5}$$

and then changing the coordinate system via the transformation

$$\boldsymbol{\alpha} = \mathbf{U}^{-1}\mathbf{x}. \tag{B.6}$$

Applying the transformation (B.6) to (B.5) yields the diagonalized system

$$\frac{d\boldsymbol{\alpha}}{dt} = \mathbf{\Lambda}\boldsymbol{\alpha} + J(t)\mathbf{U}^{-1}\mathbf{p_0}. \tag{B.7}$$

When the forcing $J(t)$ is an impulse of mass G_0, this input becomes equivalent to letting the system decay from the initial condition $\boldsymbol{\alpha}(0) = G_0\mathbf{U}^{-1}\mathbf{p_0}$, as described in appendix B.2. Thus, for an initial impulse of size G_0 system can be rewritten as

$$\frac{d\boldsymbol{\alpha}}{dt} = \mathbf{\Lambda}\boldsymbol{\alpha}, \quad \boldsymbol{\alpha}(0) = G_0\mathbf{U}^{-1}\mathbf{p_0}, \tag{B.8}$$

which has the solution

$$\alpha_i(t) = \alpha_i(0)e^{-\lambda_i t}. \tag{B.9}$$

or in matrix form

$$\boldsymbol{\alpha}(t) = e^{-\boldsymbol{\Lambda}t}\boldsymbol{\alpha}(0) = G_0 e^{-\boldsymbol{\Lambda}t}\mathbf{U}^{-1}\mathbf{p}_0. \tag{B.10}$$

The state dynamics $\mathbf{x}$ are then recovered by the eigenvector transformation $\mathbf{x} = \mathbf{U}\boldsymbol{\alpha}$,

$$\mathbf{x}(t) = G_0 \mathbf{U} e^{-\boldsymbol{\Lambda}t}\mathbf{U}^{-1}\mathbf{p}_0. \tag{B.11}$$

B.1 Expressing mass loss in a network as a sum of exponential decays

Here, we seek to express the total mass in a network as a sum of exponential decays. The mass of organic matter G remaining in the system at time t is found by summing all states $x_i(t)$

$$G(t) = \sum_i x_i(t). \tag{B.12}$$

We now project the system into the eigen-decay state co-ordinates in order to express the mass as the standard sum of exponential decays. Substituting $\mathbf{x} = \mathbf{U}\boldsymbol{\alpha}$ into (B.12) gives

$$G(t) = \sum_i \alpha_i(t) \sum_j u_{i,j} \equiv \sum_i m_i(t) \tag{B.13}$$

The effective mass m_i in the ith eigen-decay state is the weight α_i multiplied by the sum of all components of the eigenvector $\mathbf{u}_i$,

$$\mathbf{m} = \mathbf{S}\boldsymbol{\alpha} \tag{B.14}$$

where

$$\mathbf{S} = \begin{bmatrix} \sum_j u_{1j} & 0 & \dots & 0 \\ 0 & \sum_j u_{2j} & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & \sum_j u_{nj}. \end{bmatrix} \tag{B.15}$$

The relation between decaying parallel states $m_i(t)$ and the actual states $x_i(t)$ is therefore

$$\mathbf{m}(t) = \mathbf{S}\mathbf{U}^{-1}\mathbf{x}(t). \tag{B.16}$$

The matrix S simply rescales each eigenvector $\mathbf{u}_i$ by its length $\sum_j u_{ij}$. Therefore we see that by projecting the system onto eigenvectors with unit length $\sum_j u_{ij} = 1$, the total mass of the eigenstates represents the total system mass, $\sum_i x_i = \sum_i m_i$.

Multiplying both sides of (B.7) by $\mathbf{S}$, we find

$$\frac{d\mathbf{m}}{dt} = \boldsymbol{\Lambda}\mathbf{m} + J(t)\mathbf{r}, \tag{B.17}$$

where

$$\mathbf{r} = \mathbf{S}\boldsymbol{\alpha}(0) = \mathbf{S}\mathbf{U}^{-1}\mathbf{p}_0 \tag{B.18}$$

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is the initial fractionation of incoming organic matter into all eigen-decay states. Equation (B.18) is the full version of equation (A.13) from the 2x2 system.

Because Λ is diagonal, equation (B.17) represents a parallel system of exponential decays. The solution to this system for the case of an initial input or impulse of organic matter G_0 at $t = 0$ is

$$m_i(t) = G_0 r_i e^{-\lambda_i t}, \tag{B.19}$$

as derived in Appendix B.2. The total mass remaining in the system is found by summing all of the eigenstates $m_i(t)$. The fraction, $g(t) = G(t)/G_0$, of original mass remaining is

$$g(t) = \sum_i m_i(t)/G_0 \tag{B.20}$$

$$= \sum_i r_i e^{-\lambda_i t}. \tag{B.21}$$

B.2 Impulse response

When a system receives an input at $t = 0$ and no input for $t > 0$, the response of the system to this input or impulse is called the "impulse response." An impulse G_0 of mass at $t = 0$ is modeled by setting $J(t) = G_0 \delta(t)$. The immediate response of the system (B.17) is calculated by integrating both sides of (B.17) in time over the infinitesimally small duration of the impulse.

$$\int_0^{0^+} \frac{d\mathbf{m}}{dt} dt = \int_0^{0^+} \mathbf{\Lambda m} dt \int_0^{0^+} G_0 \delta(t) \mathbf{r} dt \tag{B.22}$$

Because $\mathbf{m}$ is initially the zero vector and remains finite during and after the impulse, the first term on the right hand side of (B.22) vanishes, leaving only the other two terms. Integrating both remaining terms gives

$$\mathbf{m}(0^+) - \mathbf{0} = G_0 \mathbf{r} \tag{B.23}$$

thus providing the initial condition

$$\mathbf{m}(0) = G_0 \mathbf{r} = G_0 \mathbf{S U}^{-1} \mathbf{p}_0 \tag{B.24}$$

Because $J(t) = 0$ for $t > 0$, the solution to (B.17) with the initial condition given by (B.24) is

$$m_i(t) = G_0 r_i e^{-\lambda_i t}. \tag{B.25}$$