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Scaling from trees to forests: tractable macroscopic equations for forest dynamics.
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Appendix A: Derivation of the von Foerster equation from the individual based model with the PPA

Consider the individual-based forest simulator with the PPA assumption and assume that the area of the modeled stand is infinite. Let $N_{z,i,j}(t)$ be the number of trees at time t, in some three-dimensional box of size $\triangle x$, Δy, Δz and located between heights z_1 and $z_2 = z_1 + \Delta z$, and restricted on the XY plane between x_i and $x_{i+1} = x_i + \Delta x$ and y_j and $y_{j+1} = y_j + \Delta y$. Then $N_{z,i,j}(t)$ is a random variable, whose value will vary with location. Let $\langle N_z(t)\rangle$ be an average of this random variable over the infinite number of boxes in the same vertical slice ($z_1 \leqslant z \leqslant z_2$):

$$\text{(A-1)} \qquad \langle N_z(t)\rangle = \lim_{Q\to\infty} \frac{1}{(2Q+1)^2} \sum_{i=-Q}^{Q} \sum_{j=-Q}^{Q} N_{z,i,j}(t)$$

and define $N(z,t)$ as the continuous limit:

$$\text{(A-2)} \qquad N(z,t) = \lim_{\Delta z\to 0} \frac{\langle N_z(t)\rangle}{\Delta z}.$$

Thus for small Δz, $N(z,t) \approx \langle N_z(t)\rangle \Delta z$.

Now consider changes in some time interval Δt. There are four different processes which contribute to this quantity:

1. some trees, which were in the slice at time t stay in this slice at time $t + \Delta t$;
2. some trees grow out of this slice;
3. some trees in the slice die;
4. new trees with height was less than z_1 can grow into the slice.

The following equation describes changes of $N(z,t)$ caused by these four processes:

$$\text{(A-3)} \qquad \int_{z_1}^{z_2} (N(z,t+\Delta t) - N(z,t))\, dz = \int_{z_1-\delta_1}^{z_1} N(z,t)dz - \int_{z_2-\delta_2}^{z_2} N(z,t)dz - \int_t^{t+\Delta t}\int_{z_1}^{z_2} N(z,t')\mu(z,t')dz dt',$$

where $\mu(z,t)\Delta t$ is the probability of mortality in the limit of small Δt.

The left-hand side of the equation is the change of the average number of trees in the vertical slice in the time interval Δt. The first and second terms on the RHS account for trees growing into and out of the slice respectively, and the third term represents mortality of trees in the slice. The interval δ_1, is chosen so that trees with height between $z_1 - \delta_1$ and z_1 will grow into the vertical slice during Δt:

$$\text{(A-4)} \qquad \delta_1 = \int_t^{t+\Delta t} G(z(t'),t')dt',$$

where $G(z,t)$ is the instantaneous rate of height growth, $z(t) = z_1 - \delta_1$, and $\frac{dz(x)}{dx} = G(z(x),x)$ for $t \leqslant x \leqslant t + \Delta t$.

Interval δ_2 is the corresponding height range over which trees grow out of the slice:

$$\text{(A-5)} \qquad \delta_2 = \int_t^{t+\Delta t} G(z(t'),t')dt',$$

where $z(t) = z_2 - \delta_2$, and $\frac{dz(x)}{dx} = G(z(x),x)$ for $t \leqslant x \leqslant t + \Delta t$.

Assuming small Δt we may linearize (A-4) and (A-5):

$$\text{(A-6)} \qquad \delta_1 \approx \Delta t\, G(z_1,t),$$

$$\text{(A-7)} \qquad \delta_2 \approx \Delta t\, G(z_2,t)$$

Now we can rewrite equation (A-3) as:

$$\text{(A-8)} \qquad \int_{z_1}^{z_1+\Delta z} (N(z,t+\Delta t) - N(z,t))\, dz = \Delta t\, N(z_1,t)G(z_1,t) - \Delta t\, N(z_2,t)G(z_2,t) - \int_t^{t+\Delta t}\int_{z_1}^{z_1+\Delta z} N(z,t')\mu(z,t')dz dt'$$

Linearizing the other integrals and functions in (A-8), under the assumption that Δt is small produces:

$$\text{(A-9)} \qquad \int_{z_1}^{z_1+\Delta z} \left(\Delta t \frac{\partial N(z,t)}{\partial t}\right) dz = \Delta t \left(N(z_1,t)G(z_1,t) - N(z_2,t)G(z_2,t) - \int_{z_1}^{z_1+\Delta z} N(z,t)\mu(z,t)dz\right)$$

and assuming small Δz:

$$\text{(A-10)} \qquad \frac{\partial N(z_1,t)}{\partial t} = \frac{N(z_2,t)G(z_2,t) - N(z_1,t)G(z_1,t)}{\Delta z} - N(z_1,t)\mu(z_1,t)$$

Finally, in the limit of (A-10) as $\Delta z \to 0$, we have the von Foerster equation:

$$\text{(A-11)} \qquad \frac{\partial N(z,t)}{\partial t} = -\frac{\partial N(z,t)G(z,t)}{\partial z} - N(z,t)\mu(z,t).$$